

Scalar Laplacian on Sasaki-Einstein Manifolds $Y^{p,q}$

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Abstract

We study the spectrum of the scalar Laplacian on the five-dimensional toric Sasaki-Einstein manifolds $Y^{p,q}$. The eigenvalue equation reduces to Heun's equation, which is a Fuchsian equation with four regular singularities. We show that the ground states, which are given by constant solutions of Heun's equation, are identified with BPS states corresponding to the chiral primary operators in the dual quiver gauge theories. The excited states correspond to non-trivial solutions of Heun's equation. It is shown that these reduce to polynomial solutions in the near BPS limit.

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The AdS/CFT correspondence [1] has attracted much interest as a realization of the string theory/gauge theory correspondence. It predicts that string theory in $\text{AdS}_5 \times X_5$ with X_5 be Sasaki-Einstein is dual to $\mathcal{N} = 1$ 4-dimensional superconformal field theory. Recently in [2, 3], 5-dimensional inhomogeneous toric Sasaki-Einstein manifolds $Y^{p,q}$ are explicitly constructed, besides the homogeneous manifolds, S^5 and $T^{1,1}$. The associated Calabi-Yau cones $C(Y^{p,q})$ are toric owing to the presence of the T^2 -action on the 4-dimensional Kähler-Einstein bases. Thanks to this property, the authors of [4][5] clarified the $\mathcal{N} = 1$ 4-dimensional dual superconformal field theories of IR fixed points of toric quiver gauge theories. (Further developments in this subject include [6][7].) On the other hand, in the gravity side, semiclassical strings moving on the $\text{AdS}_5 \times Y^{p,q}$ geometry are shown to be useful to establish the AdS/CFT correspondence in [8].

In this letter, we study the spectrum of the scalar Laplacian on the 5-dimensional Sasaki-Einstein manifolds $Y^{p,q}$. The eigenvalue equation of the scalar Laplacian is shown to reduce to Heun's equation[#] after the separation of variables. Heun's equation is the general second-order linear Fuchsian equation with *four* singularities. It is known that the methods to investigate hypergeometric functions with three singularities do not work for Heun's equation. Though there exist power-series solutions, the coefficients are governed by three-term recursive relations, and thus it is generally impossible to write down these series explicitly. We clarify some eigenstates of the scalar Laplacian, i.e. solutions of Heun's equation, which include BPS states dual to chiral primary operators of the superconformal gauge theory.

The metric tensor of $Y^{p,q}$ parameterized by two positive integers p, q ($p > q$) is written as [2]

$$ds^2 = \frac{1-y}{6} (d\theta^2 + \sin^2 \theta d\phi^2) + \frac{1}{w(y)q(y)} dy^2 + \frac{q(y)}{9} (d\psi - \cos \theta d\phi)^2 + w(y) [d\alpha + f(y)(d\psi - \cos \theta d\phi)]^2, \quad (1)$$

with

$$w(y) = \frac{2(b-y^2)}{1-y}, \quad q(y) = \frac{b-3y^2+2y^3}{b-y^2}, \quad f(y) = \frac{b-2y+y^2}{6(b-y^2)}, \\ b = \frac{1}{2} - \frac{p^2-3q^2}{4p^3} \sqrt{4p^2-3q^2}. \quad (2)$$

The coordinates $\{y, \theta, \phi, \psi, \alpha\}$ have the following ranges:

$$y_1 \leq y \leq y_2, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi, \quad 0 \leq \psi \leq 2\pi, \quad 0 \leq \alpha \leq 2\pi l. \quad (3)$$

[#]In [9], Heun's equation corresponding to the scalar Laplacian on the inhomogeneous manifolds constructed in [10] is examined.

The boundaries $y = y_1, y_2$ are given by the two smallest roots of the cubic $b - 3y^2 + 2y^3$,

$$y_{1,2} = \frac{1}{4p} \left(2p \mp 3q - \sqrt{4p^2 - 3q^2} \right) , \quad (4)$$

respectively, while the remaining root takes the value

$$y_3 = \frac{3}{2} - (y_1 + y_2) = \frac{1}{2} + \frac{\sqrt{4p^2 - 3q^2}}{2p} . \quad (5)$$

The period of α is $2\pi l$ with

$$l = \frac{q}{3q^2 - 2p^2 + p\sqrt{4p^2 - 3q^2}} . \quad (6)$$

For the metric (1) we have the scalar Laplacian[‡],

$$\begin{aligned} \square = & \frac{1}{1-y} \frac{\partial}{\partial y} (1-y) w(y) q(y) \frac{\partial}{\partial y} \\ & + \left(\frac{3}{2} \hat{Q}_R \right)^2 + \frac{1}{w(y)q(y)} \left(\frac{\partial}{\partial \alpha} + 3y \hat{Q}_R \right)^2 + \frac{6}{1-y} \left[\hat{K} - \left(\frac{\partial}{\partial \psi} \right)^2 \right] . \end{aligned} \quad (7)$$

The operator \hat{Q}_R represents the Reeb Killing vector field,

$$\hat{Q}_R = 2 \frac{\partial}{\partial \psi} - \frac{1}{3} \frac{\partial}{\partial \alpha} , \quad (8)$$

which corresponds to the \mathcal{R} -symmetry of the dual gauge theory [5]. The operator \hat{K} is the second Casimir of $SU(2)$, which is a part of the isometry $SU(2) \times U(1)^2$,

$$\hat{K} = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \left(\frac{\partial}{\partial \phi} + \cos \theta \frac{\partial}{\partial \psi} \right)^2 + \left(\frac{\partial}{\partial \psi} \right)^2 . \quad (9)$$

Owing to the isometry, the eigenfunction takes the form

$$\Phi(y, \theta, \phi, \psi, \alpha) = \exp \left[i \left(N_\phi \phi + N_\psi \psi + \frac{N_\alpha}{l} \alpha \right) \right] R(y) \Theta(\theta) \quad (10)$$

with $N_\phi, N_\psi, N_\alpha \in \mathbb{Z}$. Then the equation, $\square \Phi = -E \Phi$, reduces to

$$\hat{K} \Theta(\theta) = -J(J+1) \Theta(\theta) , \quad (11)$$

and

$$\begin{aligned} & \frac{1}{1-y} \frac{d}{dy} \left[(1-y) w(y) q(y) \frac{d}{dy} R(y) \right] - \left[\left(\frac{3}{2} Q_R \right)^2 + \right. \\ & \left. \frac{1}{w(y)q(y)} \left(\frac{N_\alpha}{l} + 3y Q_R \right)^2 + \frac{6}{1-y} (J(J+1) - N_\psi^2) - E \right] R(y) = 0 . \end{aligned} \quad (12)$$

[‡]The spectrum of the scalar Laplacian on $T^{p,q}$ is examined in [11]. For that of the scalar Laplacian on the Calabi-Yau cone $C(Y^{p,q})$, some properties are studied in [7].

The quantum numbers J and $Q_R = 2N_\psi - \frac{1}{3l}N_\alpha$ correspond to $SU(2)$ -spin and \mathcal{R} -charge, respectively. The regular solutions of the first equation (11) are given by Jacobi polynomials. After some calculations we find that the second equation (12) is of Fuchsian-type with four regular singularities at $y = y_1, y_2, y_3$ and ∞ , i.e. Heun's equation;

$$\frac{d^2}{dy^2}R + \left(\sum_{i=1}^3 \frac{1}{y - y_i} \right) \frac{d}{dy}R + v(y)R = 0 \quad , \quad (13)$$

where

$$v(y) = \frac{1}{H(y)} \left[\mu - \frac{y}{4}E - \sum_{i=1}^3 \frac{\alpha_i^2 H'(y_i)}{y - y_i} \right] , \quad H(y) = \prod_{i=1}^3 (y - y_i) ,$$

$$\mu = \frac{E}{4} - \frac{3}{2}J(J+1) + \frac{3}{32} \left(\frac{2N_\alpha}{3l} - Q_R \right)^2 \quad (14)$$

and

$$\alpha_1 = \pm \frac{1}{4} \left[N_\alpha \left(p + q - \frac{1}{3l} \right) - Q_R \right] \quad , \quad (15)$$

$$\alpha_2 = \pm \frac{1}{4} \left[N_\alpha \left(p - q + \frac{1}{3l} \right) + Q_R \right] \quad , \quad (16)$$

$$\alpha_3 = \pm \frac{1}{4} \left[N_\alpha \left(\frac{-2p^2 + q^2 + p\sqrt{4p^2 - 3q^2}}{q} - \frac{1}{3l} \right) - Q_R \right] \quad . \quad (17)$$

The exponents at the regular singularities are given by $\pm\alpha_i$ at $y = y_i$ ($i = 1, 2, 3$), while $-\lambda$ and $\lambda + 2$ at $y = \infty$, where we put

$$E = 4\lambda(\lambda + 2). \quad (18)$$

It is convenient to transform the singularities from $\{y_1, y_2, y_3, \infty\}$ to $\{0, 1, a = \frac{y_1 - y_3}{y_1 - y_2}, \infty\}$. This is achieved by the transformation

$$x = \frac{y - y_1}{y_2 - y_1} \quad (19)$$

together with the rescaling

$$R = x^{|\alpha_1|} (1 - x)^{|\alpha_2|} (a - x)^{|\alpha_3|} h(x) \quad , \quad (20)$$

which transforms (13) to the standard form of Heun's equation:

$$\frac{d^2}{dx^2}h(x) + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-a} \right) \frac{d}{dx}h(x) + \frac{\alpha\beta x - k}{x(x-1)(x-a)}h(x) = 0 \quad . \quad (21)$$

Here, Heun's parameters are given by

$$\begin{aligned}\alpha &= -\lambda + \sum_{i=1}^3 |\alpha_i|, \quad \beta = 2 + \lambda + \sum_{i=1}^3 |\alpha_i|, \\ \gamma &= 1 + 2|\alpha_1|, \quad \delta = 1 + 2|\alpha_2|, \quad \epsilon = 1 + 2|\alpha_3|,\end{aligned}\tag{22}$$

and k , which is called as the “accessory” parameter, is

$$\begin{aligned}k &= (|\alpha_1| + |\alpha_3|)(|\alpha_1| + |\alpha_3| + 1) - |\alpha_2|^2 \\ &\quad + a \{ (|\alpha_1| + |\alpha_2|)(|\alpha_1| + |\alpha_2| + 1) - |\alpha_3|^2 \} - \tilde{\mu}\end{aligned}\tag{23}$$

with

$$\begin{aligned}\tilde{\mu} &= -\frac{1}{y_1 - y_2}(\mu - y_1\lambda(\lambda + 2)) \\ &= \frac{p}{q} \left[\frac{2}{3}(1 - y_1)\lambda(\lambda + 2) - J(J + 1) + \frac{1}{16} \left(\frac{2}{3} \frac{N_\alpha}{l} - Q_R \right)^2 \right],\end{aligned}\tag{24}$$

$$a = \frac{1}{2} \left(1 + \frac{\sqrt{4p^2 - 3q^2}}{q} \right).\tag{25}$$

Note that the parameter a satisfies the inequality $a > 1$ reflecting $p > q$.

The regularity of the eigenfunction in the range $0 \leq x \leq 1$, which corresponds to $y_1 \leq y \leq y_2$ in the original variable, requires that $h(x)$ should be a Heun function in the sense of [12]; $h(x)$ has the zero exponents at the both boundaries $x = 0$ and 1. Simple such an example is a polynomial solution. The polynomial condition on Heun's parameters is non-trivial (see (27)), although the necessary condition is simply given by $\alpha \in \mathbb{Z}^- = \{0, -1, -2, -3, \dots\}$ (or $\beta \in \mathbb{Z}^-$). If we write the regular local solution around $x = 0$ as

$$h(x) \simeq \sum_{n=0}^{\infty} c_n x^n,\tag{26}$$

then the radius of convergence is normally $\min(1, a)$. Since $a > 1$, the series will generically have radius of convergence 1; it will therefore only represent a local solution. If a certain condition on Heun's parameters is satisfied, the radius of convergence is increased to a , so that one can obtain a Heun function [12]. However, it is not easy to determine when the condition is satisfied. When the series breaks off at the n -th order, (26) becomes a polynomial solution of degree n . Then, the coefficients c_m satisfy a set of equations,

$$\begin{aligned}-kc_0 + a\gamma c_1 &= 0, \\ P_m c_{m-1} - (Q_m + k)c_m + R_m c_{m+1} &= 0 \quad (m = 1, 2, \dots, n-1), \\ P_n c_{n-1} - (Q_n + k)c_n &= 0,\end{aligned}\tag{27}$$

where

$$\begin{aligned}
P_m &= (m-1+\alpha)(m-1+\beta) , \\
Q_m &= m((m-1+\gamma)(1+a) + a\delta + \epsilon) , \\
R_m &= (m+1)(m+\gamma)a .
\end{aligned} \tag{28}$$

First, we consider states with the quantum charges $\{J, Q_R, N_\alpha\}$ corresponding to the chiral operators of dual superconformal field theory. Then, Heun's parameters are completely fixed except for one parameter λ . The basic chiral primary operators of mesons are given by the three fields $\{\mathcal{S}, \mathcal{L}_+, \mathcal{L}_-\}$ with charges given in Table 1 [5, 8]. One can

| Meson | J | Q_R | N_α |
|-----------------|-----------------|----------------------|------------|
| \mathcal{S} | 1 | 2 | 0 |
| \mathcal{L}_+ | $\frac{p+q}{2}$ | $p+q - \frac{1}{3l}$ | 1 |
| \mathcal{L}_- | $\frac{p-q}{2}$ | $p-q + \frac{1}{3l}$ | -1 |

Table 1: Charge assignments for chiral primary operators. N_α corresponds to the $U(1)$ flavor charge divided by p .

read off these charges from the data of $Y^{p,q}$ quiver gauge theory. Note that the \mathcal{R} -charges of \mathcal{L}_\pm appear in the exponents $\alpha_{1,2}$ (see (15) and (16)) at the regular singularities $y_{1,2}$, respectively. This is consistent with the semi-classical analysis [8].

When we choose the quantum charges $\{J, Q_R, N_\alpha\}$ as Table 1 together with $\lambda = \lambda_0$ in Table 2, then Heun's parameters α and k vanish, so that Heun's equation admits a constant solution (a polynomial solution of degree 0) for each meson. These solutions represent ground states with fixed $\{J, Q_R, N_\alpha\}^b$.

| Meson | λ_0 |
|-----------------|---|
| \mathcal{S} | $\frac{3}{2}$ |
| \mathcal{L}_+ | $\frac{p}{4} \left(\frac{2p - \sqrt{4p^2 - 3q^2}}{q} + 3 \right)$ |
| \mathcal{L}_- | $\frac{p}{4} \left(\frac{-2p + \sqrt{4p^2 - 3q^2}}{q} + 3 \right)$ |

Table 2: The corresponding ground energy $E_0 = 4\lambda_0(\lambda_0 + 2)$.

^bThese constant solutions correspond to the holomorphic functions on the Calabi-Yau cone $C(Y^{p,q})$ examined in [7]. In fact, the y -dependence of the holomorphic functions, $(y-y_1)^{|\alpha_1|}(y-y_2)^{|\alpha_2|}(y-y_3)^{|\alpha_3|}$, can be read off from the equation (20). We thank Christopher Herzog for pointing out [7].

The conformal dimension Δ of the dual operator is related to the ground energy E_0 by the formula [1][11],

$$\Delta = -2 + \sqrt{4 + E_0} . \quad (29)$$

Using Table 2, we obtain the equality $\Delta = \frac{3}{2}Q_R$. In this way, we have reproduced the BPS condition of the dual operator from the gravity side (see also [7]). This is consistent with the AdS/CFT correspondence.

Next, let us consider the first excited state with $\lambda = \lambda_0 + 1$ and $\{J, Q_R, N_\alpha\}$ in Table 1. We find that it exists and the corresponding Heun function $h_1(x)$ is given by the polynomial solution of degree 1;

$$h_1(x) = \begin{cases} 1 - \frac{2p+3q+\sqrt{4p^2-3q^2}}{q+\sqrt{4p^2-3q^2}}x & \text{for } \mathcal{S} , \\ 1 + \frac{2p-3q-pq+(1+p)\sqrt{4p^2-3q^2}}{(1+p)(-q+\sqrt{4p^2-3q^2})}(x-1) & \text{for } \mathcal{L}_+ , \\ 1 - \frac{2p+3q+pq+(1+p)\sqrt{4p^2-3q^2}}{(1+p)(q+\sqrt{4p^2-3q^2})}x & \text{for } \mathcal{L}_- . \end{cases} \quad (30)$$

The existence of the higher excited states with $\lambda = \lambda_0 + n$ ($n = 2, 3, \dots$) is indicated by numerical simulation. For them, Heun functions are not polynomials, although the parameter α is a negative integer $-n$. The difference $\Delta_n - \frac{3}{2}Q_R$ is equal to $2n$, where $\Delta_n = -2 + \sqrt{4 + E_n}$ and $E_n = 4(\lambda_0 + n)(\lambda_0 + n + 2)$.

We find more general solutions with Heun's parameters summarized in Table 3. These

| | α | β | γ | δ | ϵ |
|-----|----------|--|----------|----------|---|
| A | $-n$ | $2 + 3N + n$ | $1 + N$ | $1 + N$ | $1 + N$ |
| B | $-n$ | $2 + n + Np + \frac{Np(2p+q-\sqrt{4p^2-3q^2})}{2q}$ | 1 | $1 + Np$ | $1 + \frac{Np(2p+q-\sqrt{4p^2-3q^2})}{2q}$ |
| C | $-n$ | $2 + n + Np + \frac{Np(-2p+q+\sqrt{4p^2-3q^2})}{2q}$ | $1 + Np$ | 1 | $1 + \frac{Np(-2p+q+\sqrt{4p^2-3q^2})}{2q}$ |

| | k | λ |
|-----|--|---|
| A | $-\frac{n(2+n+3N)(2p+3q+\sqrt{4p^2-3q^2})}{6q}$ | $n + \frac{3}{2}N$ |
| B | $-n\frac{6Np^2+2p(2+n+3Nq)+(2+n)(3q+\sqrt{4p^2-3q^2})}{6q}$ | $n + \frac{p}{4}\left(\frac{2p-\sqrt{4p^2-3q^2}}{q} + 3\right)N$ |
| C | $-n\frac{3Npq+3q(n+2)+4p+8n+(2+n+3Npq)\sqrt{4p^2-3q^2}}{6q}$ | $n + \frac{p}{4}\left(\frac{-2p+\sqrt{4p^2-3q^2}}{q} + 3\right)N$ |

Table 3: Heun's parameters and λ for three-types of states, A , B and C ($N = 1, 2, \dots$ and $n = 0, 1, 2, \dots$).

solutions reduce to those given above if we set $N = 1$. Even if $N > 1$, one can obtain the

similar results to those found in the case of $N = 1$. Especially, when we set $n = 0$, the states A , B and C represent those dual to N mesonic chiral operators corresponding to \mathcal{S} , \mathcal{L}_+ and \mathcal{L}_- , respectively. The quantum charges for N meson operators are obtained by multiplying charges in Table 1 and 2 by N ,

$$J \rightarrow NJ, \quad Q_R \rightarrow NQ_R, \quad N_\alpha \rightarrow NN_\alpha \quad \text{and} \quad \lambda_0 \rightarrow N\lambda_0. \quad (31)$$

Finally, taking N large, we find that the Heun functions take the form $h_n(x) \simeq n$ -polynomial + $\mathcal{O}(1/N)$, and in the limit $N \rightarrow \infty$, they tend to the following polynomials;

$$h_n(x) \rightarrow \begin{cases} h_1(x)^n & \text{for } A \\ x^n & \text{for } B \\ (1-x)^n & \text{for } C \end{cases} \quad (32)$$

where $h_1(x)$ is defined in (30). These polynomials are derived by using $1/N$ -expansion of three-term recursions (27)[◊]. The near BPS states with $n \ll N$ correspond to the near BPS geodesics in [8].

In this letter, we studied the spectrum of the scalar Laplacian on $Y^{p,q}$ corresponding to mesonic chiral operators of the dual superconformal field theories. The states dual to the baryonic chiral operators will be extracted in the similar manner. In [13], inhomogeneous toric Sasaki-Einstein manifolds in arbitrary odd-dimensions have been constructed. It is also interesting to examine the spectrum of the scalar Laplacian on seven-dimensional toric Sasaki-Einstein manifolds and the 3-dimensional dual gauge theories. For a large class of Sasaki-Einstein spaces, their volumes are calculated in [14]. It is interesting to derive the volume of the non-toric 3-Sasakian manifolds constructed in [15]. We hope to report this point in the future. After [16], many compact Sasaki-Einstein manifolds [17] have been derived from AdS-Kerr black holes. It is important to clarify the spectrum on these manifolds. These are left for future investigations.

We are grateful to Yoshitake Hashimoto and Sean Hartnoll for discussions and useful comments. This work is supported by the 21 COE program ‘‘Constitution of wide-angle mathematical basis focused on knots’’ and in part by the Grant-in Aid for scientific Research (No. 17540262 and No. 17540091) from Japan Ministry of Education.

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